FINITE DIFFERENCES WITH ONE VARIABLE

Definition of Finite Differences:

1) $\Delta f(x) = \dfrac{f(x+1)-f(x)}{1} = f(x+1)-f(x)$ **2)** $\Delta^{n+1} f(x) = \Delta(\Delta^n f(x))$

Example:

If $f(x) = x^2 + 3x$ then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 + 3(x+1) - x^2 - 3x = 2x + 4$$

$$\Delta^2 f(x) = \Delta(\Delta f(x) = \Delta(2x+4) = [2(x+1)+4] - [2x+4] = 2$$

Definition of factorial exponents:

$x^{(n)} = x(x-1)(x-2)...(x-n+1)$ if $n \geq 1$

$x^{(0)} = 1$

Examples: **1)** $x^{(5)} = x(x-1)(x-2)...(x-5+1) = x(x-1)(x-2)(x-3)(x-4)$ (5 factors)

2) $7^{(3)} = 7(7-1)(7-2) = 210$

3) $1^{(n)} = 0$ for all n greater than 1.

Theorems:

1) *If* $f(x) = x^{(n)}$ then $\Delta f(x) = nx^{(n-1)}$

Proof:

$$f(x) = x^{(n)} \Rightarrow \Delta f(x) = [x+1]^{(n)} - [x]^{(n)} = [(x+1)x(x-1)...(x-n+2)] - [x(x-1)...(x-n+2)(x-n+1)]$$
$$= x(x-1)...(x-n+2)[(x+1)-(x-n+1)] = x^{(n-1)}[n]$$

2) $\Delta[kf(x)] = k[\Delta f(x)]$

Proof:

$$\Delta[kf(x)] = kf(x+1) - kf(x) = k[f(x+1) - f(x)] = k[\Delta f(x)]$$

3) $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

Proof:

$$\Delta[f(x)+g(x)] = [f(x+1)+g(x+1)] - [f(x)+g(x)]$$
$$= [f(x+1)-f(x)] + [g(x+1)-g(x)] = \Delta f(x) + \Delta g(x)$$

The proof for $\Delta[f(x)-g(x)]$ is similar.

4) *If* f and g are functions defined on the integers and $\Delta f(x) = \Delta g(x)$ then $f(x) = g(x) + c$.

Lemma: If h(x) is defined on the integers and Δh(x)=0 for all x, then h(x) = c, a constant.

Proof:

$\Delta h(x) = h(x+1) - h(x) = 0 \Rightarrow h(x+1) = h(x)$ for all integers x $\Rightarrow h(x) = c$.

Proof of the Theorem:

$$\Delta f(x) = \Delta g(x) \Rightarrow \Delta f(x) - \Delta g(x) = 0 \Rightarrow \Delta[f(x)-g(x)] = 0 \Rightarrow f(x) - g(x) = c.$$

5) $\Delta a^x = a^x(a-1)$

Proof:

$$\Delta a^x = a^{x+1} - a^x = a^x(a-1)$$

Definition of Finite Antidifferences:

$\sum f(x) = F(x)$ if and only if $\Delta F(x) = f(x)$.

Note: Because of Theorem 4, if two function F(x) and G(x) are both finite antidifferences of f(x) then $\Delta F(x) = f(x) = \Delta G(x) \Rightarrow F(x) = G(x) + c$. Hence antidifferences are actually groups of functions written as "[an expression involving x] + c", where the difference between one member of the group and another is in the value of c.

Theorem 6

$$\sum a^x = \frac{a^x}{a-1} + c$$

Proof:

$$\Delta\left[\frac{a^x}{a-1} + c\right] = \frac{1}{a-1}\Delta a^x + \Delta c = \frac{1}{a-1}a^x(a-1) + 0 = a^x$$

Definition of Summation:

$$\sum_{x=a}^{b} f(x) = f(a) + f(a+1) + f(a+2) + \ldots + f(b-1) + f(b)$$

Theorem 7. The Fundamental Theorem of Summations

$$\sum_{x=a}^{b} f(x) = F(b+1) - F(a) = \left[F(x)\right]_a^{b+1} \quad \text{where } \Delta F(x) = f(x). \quad \text{(Note: } \sum f(x) = F(x) \text{.)}$$

Proof:

$$\sum_{x=a}^{b} f(x) = \sum_{x=a}^{b} \Delta F(x)$$

$$= [F(a+1) - F(a)] + [F(a+2) - F(a+1)] + [F(a+3) - F(a+2)] + \ldots + [F(b+1) - F(b)]$$

$$= F(b+1) - F(a)$$

Note: The F(a+1) of the first group cancels out with the F(a+1) of the second group etc. The only terms that do not cancel out are the -F(a) from the first group and the F(b+1) from the last group.

Applications:

1) Find a formula for a sequence such as: $a_n =$ 0 5 22 57 116 205 ...

Solution;

$\Delta a_n = 5 \quad 17 \quad 35 \quad 59 \quad 89 \quad \ldots$

$\Delta^2 a_n = 12 \quad 18 \quad 24 \quad 30 \quad \ldots$

$\Delta^3 a_n = 6 \quad 6 \quad 6 \quad \ldots$

Because $\Delta^3 a_n = 6, \sum \Delta^3 a_n = \Delta^2 a_n = 6n^{(1)} + c_1$.

Since $\Delta^2 a_1 = 12,\ 12 = 6(1)^{(1)} + c_1$ and $c_1 = 6$.

Because $\Delta^2 a_n = 6n^{(1)} + 6, \sum \Delta^2 a_n = \Delta a_n = 3n^{(2)} + 6n^{(1)} + c_2$.

Since $\Delta a_1 = 5,\ 5 = 3(1)^{(2)} + 6(1)^{(1)} + c_2$ and $c_2 = -1$.

Finally $a_n = n^{(3)} + 3n^{(2)} - n^{(1)} + c_3$ with $a_1 = 0$.

Since $a_1 = 0 = 1^{(3)} + 3(1)^{(2)} - 1^{(1)} + c_3$, $c_3 = 1$.

Thus $a_n = n^{(3)} + 3n^{(2)} - n^{(1)} + 1 = n^3 - 2n + 1$.

2) Find a formula for a sum such as: $\sum_{t=1}^{n} t^2$

Solution:

1) Represent t^2 in terms of t factorials.

Use synthetic division to divide by t-0, then t-1, then t-2 etc.

$$\begin{array}{r|ccc} 0 & 1 & 0 & 0 \\ & & 0 & 0 \\ \hline & 1 & 0 & 0 \end{array}$$

implies constant term = R = 0

$$\begin{array}{r|cc} 1 & 1 & 0 \\ & & 1 \\ \hline & 1 & 1 \end{array}$$

implies first degree coefficient = R = 1

$$\begin{array}{r|c} 2 & 1 \\ & \\ \hline & 1 \end{array}$$

implies second degree coefficient = R = 1

Thus $t^2 = t^{(2)} + t^{(1)} + 0$

2)

$$\sum_{t=1}^{n} t^2 = \sum_{t=1}^{n} t^{(2)} + t^{(1)} = \sum_{t=1}^{n} \Delta\left[\frac{t^{(3)}}{3} + \frac{t^{(2)}}{2}\right] = \frac{(n+1)^{(3)}}{3} + \frac{(n+1)^{(2)}}{2} - \frac{1^{(3)}}{3} - \frac{1^{(2)}}{2}$$

$$= (n+1)^{(2)}\left[\frac{n-1}{3} + \frac{1}{2}\right] = (n+1)n\left[\frac{2n+1}{6}\right] = \frac{n(n+1)(2n+1)}{6}$$

3) Find a formula for the sequence a_n = 18 52 146 420 1234

Solution:

a_n	18		52		146		420		1234
Δa_n		34		94		274		814	
$\Delta^2 a_n$			60		180		540		

We notice that 60 * 3 = 180 and 180 * 3 =540.

It follows that $\Delta^2 a_n = A \cdot 3^n$ where A is some constant. We note that A = 20 works.

Then

$$\Delta^2 a_n = 20 \cdot 3^n$$

$$\sum \Delta^2 a_n = \Delta a_n = \frac{20 \cdot 3^n}{3-1} + c_1 = 10 \cdot 3^n + c_1$$

$$\Delta a_1 = 34 = 10 \cdot 3^1 + c_1 \Rightarrow c_1 = 4$$

$$\sum \Delta a_n = a_n = \sum 10 \cdot 3^n + 4 = \frac{10 \cdot 3^n}{3-1} + 4n^{(1)} + c_2 = 5 \cdot 3^n + 4n + c_2$$

$$a_1 = 18 = 5 \cdot 3^1 + 4 \cdot 1 + c_2 \Rightarrow c_2 = -1$$

Thus, $a_n = 5 \cdot 3^n + 4n - 1$

4) Find an explicit formula for $a + ar + ar^2 + ar^3 + \ldots ar^{n-1} = \sum_{x=0}^{n-1} ar^x$

Solution:

$$\sum_{x=0}^{n-1} ar^x = \left[\frac{ar^x}{r-1}\right]_0^{n-1+1} = \frac{ar^n - ar^0}{r-1} = \frac{a(r^n - 1)}{r-1}$$

FINITE DIFFERENCES WITH TWO VARIABLES

Definition:

If f(x,y) is a function of two variables then

$$\Delta_x f(x,y) = f(x+1,y) - f(x,y) \quad \text{and} \quad \Delta_y f(x,y) = f(x,y+1) - f(x,y)$$

Similarly, $\sum_x f(x,y) = F(x,y)$ iff $\Delta_x F(x,y) = f(x,y)$ and

$\sum_y f(x,y) = F(x,y)$ iff $\Delta_y F(x,y) = f(x,y)$

Applications:

1) Find a function f(x,y) which has the following array of values.

Y=4	69	76	85	96	109	124	141
Y=3	31	37	45	55	67	81	97
Y=2	11	16	23	32	43	56	71
Y=1	3	7	13	21	31	43	57
X=	1	2	3	4	5	6	7

Using differences in the row Y = 1, yields f(x,1) = $x^2 + x + 1$

Similarly f(x,2) = $x^2 + 2x + 8$

f(x,3) = $x^2 + 3x + 27$

f(x,4) = $x^2 + 4x + 64$

and f(x,5) = $x^2 + 5x + 125$

Applying differences to these yields

$$\begin{array}{llll}
x^2 + 5x + 125 & & & \\
 & x + 61 & & \\
x^2 + 4x + 64 & & 24 & \\
 & x + 37 & & 6 \\
x^2 + 3x + 27 & & 18 & \\
 & x + 19 & & 6 \\
x^2 + 2x + 8 & & 12 & \\
 & x + 7 & & \\
x^2 + x + 1 & & &
\end{array}$$

Thus

$\Delta_y^3 f = 6 \Rightarrow \Delta_y^2 f = 6y + c_1$ and $\Delta_y^2 f(x,1) = 12 = 6y + c_1 \Rightarrow \Delta_y^2 f(x,y) = 6y + 6$

$\Delta_y f(x,y) = 3y^{(2)} + 6y + c_2$ and $\Delta_y f(x,1) = x + 7 = 3(1)^{(2)} + 6(1) + c_2 \Rightarrow \Delta_y f(x,y) = 3y^{(2)} + 6y + x + 1$

$f(x,y) = y^{(3)} + 3y^{(2)} + xy + 7y + c_3$ and $f(x,1) = x^2 + x + 1 = (1)^{(3)} + 3(1)^{(2)} + x(1) + 1(1) + c_3$

yields $f(x,y) = y^{(3)} + 3y^{(2)} + xy + 7y + x^2 = y(y-1)(y-2) + 3y(y-1) + xy + 7y + x^2 = y^3 + xy + x^2$

2) Consider the following double summation.

$$\sum_{x=1}^{4}\sum_{y=2}^{7}[xy - x^{(2)} + 2y - 5] = \sum_{x=1}^{4}\left[\frac{xy^{(2)}}{2} - x^{(2)}y + y^{(2)} - 5y\right]_2^8$$

$$= \sum_{x=1}^{4}[(28x - 8x^{(2)} + 56 - 40) - (x - 2x^{(2)} + 2 - 10)]$$

$$= \sum_{x=1}^{4}[-6x^{(2)} + 27x + 24]$$

$$= \left[-2x^{(3)} + \frac{27x^{(2)}}{2} + 24x\right]_1^5$$

$$= (-120 + 270 + 120) - (0 + 0 + 24)$$

$$= 246$$

USING PARAMETERS TO FIND y = f(x)

Suppose you are given a table of (x,y) values in which the increments in x are not a constant like in 1, 2, 3, … for a_n.

Example:

x	-2	-1	1	5	13	29	61
y	0	0	2	6	12	20	30

How could you find y = f(x)?

Solution:

Step 1: Use finite differences to find x and y as sequences in terms of t.

Using finite differences on the sequence of x's above we would get $x = 2^{t-1} - 3$ = g(t).

Similarly using finite differences on the sequence of y's yields $y = t^2 - 3t + 2$ = h(t).

Step 2:

Then $t = g^{-1}(x) = \dfrac{\log(x+3)}{\log 2} + 1$ and y = h(t) = $\left[\dfrac{\log(x+3)}{\log 2} + 1\right]^2 - 3\left[\dfrac{\log(x+3)}{\log 2} + 1\right] + 2$

RECURSION EQUATIONS

Definitions:

1) A **recursion equation of order n** is an equation that can be written in the form
$a_n f(x+n)+a_{n-1}f(x+n-1)+a_{n-2}f(x+n-2)+...+a_0 f(x)=p(x)$, *where* $a_1,a_2,...a_n$ are constants
2) A **homogeneous recursion equation of order n** is one for which the p(x) in the above equation is equal to zero.
3) **Notation:** If f(x) is a solution of a recursion equation of order n then **the homogeneous solution of that recursion equation,** $f_H(x)$, is actually the set of functions that are solutions of the associated homogeneous equation.
$f_P(x)$ represents any one **particular solution** of the original equation.
Representations of $f_H(x)$ include constants that change from one member of the set to another .
(Note: In Theorem 4, $f_H(x)=\{c\}$.)
4) A **boundary value condition of a recursion equation** is a specific value of any function that is a solution of the recursion equation.
5) The **characteristic equation** associated with the recursion equation
$a_n f(x+n)+a_{n-1}f(x+n-1)+...+a_0 f(x)=p(x)$ is the equation
$a_n\lambda^n+a_{n-1}\lambda^{n-1}+...+a_0=0$ where λ is a complex number.

Theorem 8. If f(x) and g(x) are solutions of the same recursion equation of order n then f(x) – g(x) is a member of $f_H(x)$.

Proof:

If $a_n f(x+n)+a_{n-1}f(x+n-1)+...+a_0 f(x)=p(x)$
and $a_n g(x+n)+a_{n-1}g(x+n-1)+...+a_0 g(x)=p(x)$
then $a_n f(x+n)+...+a_0 f(x)=a_n g(x+n)+...+a_0 g(x)$

$$a_n[f(x+n)-g(x+n)]+a_{n-1}[f(x+n-1)-g(x+n-1)]+...+a_0[f(x)-g(x)]=0$$

Thus f(x)-g(x) is a homogeneous solution of the equation.

Comment:

Recursion equations, without boundary value conditions, generally have more than one solution. The set of solutions of a recursion equation is called the **general solution of the equation.** General solutions of recursion equations can always be written in the form

$$f(x)=f_P(x)+f_H(x)=\{f_P(x)+f_c(x)\mid f_c(x)\in f_H(x)\}$$

It is easy to see that all such functions are solutions of the recursion equation because

$$\begin{aligned}&a_n f(x+n)+a_{n-1}f(x+n-1)+...+a_0 f(x)\\ &=a_n[f_p(x+n)+f_c(x+n)]+a_{n-1}[f_p(x+n-1)+f_c(x+n-1)]+...+a_0[f_p(x)+f_c(x)]\\ &=\{a_n f_p(x+n)+a_{n-1}f_p(x+n-1)+...+a_0 f_p(x)\}+\{a_n f_c(x+n)+a_{n-1}f_c(x+n-1)+...+a_0 f_c x)\}\\ &=p(x)+0\end{aligned}$$

In effect, the $f_P(x)$ and $f_H(x)$ are **superimposed on top of each other.**
When a recursion equation is accompanied by a sufficient number of boundary value conditions, one is able to determine which member of f(x) is **<u>the solution of the problem.</u>**

Definition:

A **linear combination of a set of functions,** $\{f_1, f_2, ... f_n\}$ is any expression of the form

$$c_1 f_1(x) + c_2 f_2(x) + ... c_n f_n(x).$$

Typically every member of $f_H(x)$ can be written as a linear combination of some subset of $f_H(x)$. The number of members in the smallest subset of $f_H(x)$ that can be used to represent every member of $f_H(x)$ is **the dimension of** $f_H(x)$. The members of any such minimal subset of $f_H(x)$ are said to **span** $f_H(x)$. (In Theorem 4, $f_H(x)$ ={c} and $f_H(x)$ is spanned by the one function f(x) = 1. Thus the dimension of this $f_H(x)$ is 1 because only one function is needed to span it.)

Theorem 9A. If the roots $\lambda_1, \lambda_2, ... \lambda_n$ of an nth order recursion equation's characteristic equation are distinct then the set of homogeneous functions of the recursion equation is spanned by $\{\lambda_1^x, \lambda_2^x, ... \lambda_n^x\}$.

Proof in the case when n=1.

$a_1 f(x+1) + a_0 f(x) = 0$ yields the characteristic equation $a_1\lambda + a_0 = 0$ and this yields $\lambda = -\frac{a_0}{a_1}$

$$a_1 f(x+1) + a_0 f(x) = 0 \quad \Rightarrow \quad f(x+1) = -\frac{a_0}{a_1} f(x) = \lambda f(x)$$

$$= \lambda[\lambda f(x-1)] = \lambda^2 f(x-1) = ... = \lambda^{x+1} f(x-x) = f(0)\lambda^{x+1}$$

$$\Rightarrow \quad f(x) = f(0)\lambda^x = c \cdot \lambda^x$$

The case n = 2 will be discussed later.

Finding a particular solution. (Note: All you have to do is find one. Any one will do.)

If p(x) is a polynomial there will always exist a particular solution that is of the same degree as p(x) or less. The coefficients of $f_p(x)$ can be found by equating the corresponding coefficients on both sides of the equation: $\quad a_n f_p(x+n) + a_{n-1} f_p(x+n-1) + ... + a_0 f_p(x) = p(x)$

If p(x) is an exponential function there will always exist a particular solution that is an exponential function with the same base.

Theorem 9B. Let λ_1 and λ_2 be the roots of the characteristic equation associated with the recursion equation, $f(x+2)+a_1 f(x+1)+a_0 f(x)=p(x)$ then

$$f_H(x)=\begin{cases} A\lambda_1^x+B\lambda_2^x & \text{when } \lambda_1\neq\lambda_2 \\ Ax\lambda^{x-1}+B\lambda^x & \text{when } \lambda_1=\lambda_2=\lambda \end{cases} \qquad \text{where A and B are constants.}$$

Proof:

Note that the characteristic equation is $\lambda^2+a_1\lambda+a_0=0$ and its solutions are:

$$\lambda_1=\frac{-a_1+\sqrt{a_1^2-4a_0}}{2} \quad \text{and} \quad \lambda_2=\frac{-a_1-\sqrt{a_1^2-4a_0}}{2}, \quad \text{(not necessarily distinct)}$$

Let g(x) = f(x+1) + kf(x), **(equation 1)** where k is some constant to be specified later. Then g(x+1) = f(x+2) +kf(x+1). It follows that f(x+2) = g(x+1) – kf(x+1). Substituting this into the homogeneous recursion equation we get: $[g(x+1)-kf(x+1)]+a_1 f(x+1)+a_0 f(x)=0$. Thus,

$$g(x+1)+[a_1-k]f(x+1)+a_0 f(x)=0. \text{ **(equation 2)**}$$

From equation 1 we get f(x+1) = g(x) – kf(x). Substituting this into equation 2 we get,

$g(x+1)+(a_1-k)[g(x)-f(x)]+a_0 f(x)=0$. It follows that

$$g(x+1)+(a_1-k)g(x)+[k^2-a_1k+a_0]f(x)=0.$$

This will be a first order recursion equation in g if we let k be a root of $k^2-a_1k+a_0=0$.

Thus we choose k to be $\dfrac{a_1+\sqrt{a_1^2-4a_0}}{2}=\lambda_1+a_1$.

Solving the remaining first order recursion equation in g we get:

$$\lambda_g+(a_1-k)=0 \;\Rightarrow\; \lambda_g=k-a_1=\lambda_1 \text{ and since } g_p(x)=0, \quad g(x)=A'\lambda_1^x+0=A'\lambda_1^x.$$

Substituting back into equation 1 we get :

$$f(x+1)+(\lambda_1+a_1)f(x)=A'\lambda_1^x=p(x).$$

This is a first order recursion equation in f with $f_H(x)=B(-\lambda_1-a_1)^x=B\lambda_2^x$.

Solving for $f_p(x)=C\lambda_1^x$ we get $C\lambda_1^{x+1}+(\lambda_1+a_1)C\lambda_1^x=A'\lambda_1^x$. **It follows that**

if $2\lambda_1+a_1\neq 0$ then C=$\dfrac{A'}{2\lambda_1+a_1}=A$, a constant **and** $f(x)=f_p(x)+f_H(x)=A\lambda_1^x+B\lambda_2^x$.

$f(x)=A(-1)^x+B(1)^x+C(2)^x-7$ If $2\lambda_1+a_1=0$ then $\lambda_1=\dfrac{-a_1}{2}=\lambda_2=\lambda$. In this case we find a particular solution of the type

$f_p(x)=Cx\lambda^{x-1}$. Substituting into $f(x+1)+(\lambda_1+a_1)f(x)=A'\lambda_1^x=p(x)$, we get

$C(x+1)\lambda^x+(\lambda+a_1)Cx\lambda^{x-1}=A'\lambda^x \;\Rightarrow\; Cx\lambda^{x-1}(\lambda+\lambda+a_1)+C\lambda^x=Cx\lambda^{x-1}(0)+C\lambda^x=A'\lambda^x$.

Thus C = A' = A and $f(x)=Ax\lambda^{x-1}+B\lambda^x$

Theorem 9A also works for third order recursion equations and for recursion equations in which the roots of the characteristic equation are complex numbers, as illustrated by the following examples:

Example 1:

$f(1)=2,\ f(2)=-3,\ f(3)=6$

$f(x+3)=-f(x+2)+4f(x+1)+4f(x)+7$ which has values 2, -3, 6, -3, 22, -3,...

Solution:

$\lambda^3+\lambda^2-4\lambda-4=0 \Rightarrow \lambda=-1,\ -2,\text{ or } 2$

Therefore $f_H(x)=A(-1)^x+B(-2)^x+C(2)^x$

Trying $f_p(x)=C$ we get C = -C +4 C +4 C + 7 and $f_p(x)=C=-\frac{7}{6}$

Thus, $f(x)=A(-1)^x+B(-2)^x+C(2)^x-\frac{7}{6}$.

Using the boundary conditions, we get $A=-\frac{11}{6}$, $B=-\frac{1}{3}$, and $C=\frac{1}{3}$.

In conclusion, $f(x)=-\frac{11}{6}(-1)^x-\frac{1}{3}(-2)^x+\frac{1}{3}(2)^x-\frac{7}{6}$ which yields the same sequence.

Example 2:

$f(1)=5,\ f(2)=-4$

$f(x+2)=-f(x)+x-3$ which has values 5, -4, -7, 3, 7, -2, ...

Solution:

$\lambda^2+1=0 \Rightarrow \lambda=\pm i \Rightarrow f_H(x)=A(i)^x+B(-i)^x$

$f_p(x)=ax+b \Rightarrow a(x+2)+b=-ax-b+x-3 \Rightarrow a=\frac{1}{2}$ and $b=-2$

Thus, $f(x)=A(i)^x+B(-i)^x+\frac{1}{2}x-2$.

Using the boundary conditions, we get $A=\frac{6-13i}{4}$ and $B=\frac{6+13i}{4}$.

In conclusion, $f(x)=\frac{6-13i}{4}(i)^x+\frac{6+13i}{4}(-i)^x+\frac{x}{2}-2$ which yields the same sequence

Applications

1. In an amortization problem f(x) represents the account balance at the end of x periods, f(0) = original Principle = P and $f(x+1)=(1+r/n)f(x)+d$, where r is the annual interest rate, n is the number of amortization periods per year, and d is the amount added to or removed from the account each period. Find an explicit formula for f(x).

   Solution: Let $K=\left(1+\frac{r}{n}\right)$. Then f(0) = P and f(x+1) = Kf(x)+d.

   The characteristic equation is $\lambda = K$ and $f_p(x)=c \Rightarrow f_p(x)=\frac{d}{1-K}$.

   Thus, $f(x)=A\cdot K^x+\frac{d}{1-K}$ and using the boundary condition f(0) = P we get

   $$f(x)=\left(P+\frac{d}{K-1}\right)K^x+\frac{d}{1-K}$$

2. The Fibonacci sequence is defined by f(1)=f(2)=1 and f(x+2)=f(x+1) + f(x). Find an explicit formula for this sequence.

   Solution: The characteristic equation is $\lambda^2=\lambda+1$ which has the solutions $\lambda=\frac{1\pm\sqrt{5}}{2}$.

   A particular solution is $f_p(x)=0$ and therefore $f(x)=A\left(\frac{1+\sqrt{5}}{2}\right)^x+B\left(\frac{1-\sqrt{5}}{2}\right)^x$

   Using the boundary conditions we eventually get:

   $$f(x)=\frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^x-\frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^x$$

3. Five sailors, marooned on an island, pick a bunch of coconuts and leave them overnight to be divided evenly among themselves the next day. During the night the first sailor awakens and decides to take his share. Dividing the pile by five he has an extra coconut left over which he throws to a monkey and then takes his fifth. The second sailor comes along and does the same thing. Throwing an extra coconut to the monkey and taking his fifth of the remaining pile. This continues through all five sailors. The next morning the five sailors evenly divide what is left and again find that there is one coconut left over for the monkey. What is the minimum number of coconuts that the sailors could have picked on the previous day?

4. The Tower of Hanoi.
   f(1) = 1, f(x+1)=2f(x)+1 .....

5. Find the total number of rectangles in an m x n rectangle. (Finite differences in 2D).

6. Pick's Theorem. Find a formula for the area of a polygon whose vertices are lattice points by counting the number of lattice points in its interior and the number of lattice points on its boundary.

7. The Battle of Trafalgar – Lord Nelson's Problem.

Navy 1 (hit percentage =p)	Navy 2 (hit percentage =q)
f(0) = L	g(0) = S
L-qS	S-pL

Etc.

$$f(x+1) = f(x) - qg(x) \qquad g(x+1) = g(x) - pf(x)$$

$$\Delta f(x) = -qg(x) \qquad \Delta g(x) = -pf(x)$$

$$\Delta^2 f(x) = -q\Delta g(x) \qquad \Delta^2 g(x) = -p\Delta f(x)$$

$$\Delta^2 f(x) = -q[-pf(x)] \qquad \Delta^2 g(x) = -p[-qg(x)]$$

$$\Delta^2 f(x) - pqf(x) = 0 \qquad \Delta^2 g(x) - pqg(x) = 0$$

In both cases $\lambda = \pm\sqrt{pq}$.

$$\therefore f(x) = A(1+\sqrt{pq})^x + B(1-\sqrt{pq})^x = g(x)$$

Using f(0) = L, f(1) = L - qS and g(0) = S, g(1) = S - pL, we get

For f(x), $A = \dfrac{L\sqrt{pq}\text{-}qS}{2\sqrt{pq}} \qquad B = \dfrac{L\sqrt{pq}+qS}{2\sqrt{pq}}$

For g(x), $A = \dfrac{S\sqrt{pq}-pL}{2\sqrt{pq}} \qquad B = \dfrac{S\sqrt{pq}+pL}{2\sqrt{pq}}$

If p = q and h(x) = f(x) - g(x) then

$$\Delta h(x) = \Delta f(x) - \Delta g(x) = -qg(x) - [-pf(x)] = pf(x) - qg(x) = ph(x)$$

It follows that $h(x) = C(1+p)^x = (L-S)(1+p)^x$.

We want h(x) when g(x) = 0. When p = q, we get

$$g(x) = \frac{S-L}{2}(1+p)^x + \frac{S+L}{2}(1-p)^x = 0 \Rightarrow \left(\frac{1+p}{1-p}\right)^x = \frac{L+S}{L-S} \Rightarrow x = \log\left(\frac{L+S}{L-S}\right) \div \log\left(\frac{1+p}{1-p}\right) = x_0$$

Case 1: L=100, S=80, p=.1 $\Rightarrow x_0 = 10.95 \Rightarrow h(x_0) = 56.8$

Case 2: L=1300, S=1200, p=.01 $\Rightarrow x_0 = 160.94 \Rightarrow h(x_0) = 496$

Note: L, S and $h(x_0)$ nearly fit the Pythagorean theorem (especially when p<<1).

The Geistfeld Transform

Definition: $A(s)=\mathcal{G}\{a_n\}=\sum_{n=1}^{\infty}s^{-n}a_n$

Theorems:

$$\mathcal{G}\{0\}=\sum_{n=1}^{\infty}s^{-n}\cdot 0=0$$

$$\mathcal{G}\{\lambda^n\}=\sum_{n=1}^{\infty}s^{-n}\lambda^n=\frac{s^{-1}\lambda}{1-s^{-1}\lambda}=\frac{\lambda}{s-\lambda}$$

$$\mathcal{G}\{(n-1)\lambda^n\}=\sum_{n=1}^{\infty}s^{-n}(n-1)\lambda^n=\sum_{n=1}^{\infty}u^n(n-1)\quad \text{(Let u} = s^{-1}\lambda\text{).}$$

$$=u^2\sum_{n=1}^{\infty}(n-1)u^{n-2}=u^2\left(\sum_{n=1}^{\infty}u^{n-1}\right)'=u^2\left(\frac{1}{1-u}\right)'=u^2\left(\frac{1}{(1-u)^2}\right)=\frac{(s^{-1}\lambda)^2}{(1-s^{-1}\lambda)^2}=\frac{\lambda^2}{(s-\lambda)^2}$$

$$\mathcal{G}\{(n-1)(n-2)\lambda^n\}=\sum_{n=1}^{\infty}s^{-n}\lambda^n(n-1)(n-2)=u^3\sum_{n=1}^{\infty}(n-1)(n-2)u^{n-3}\quad \text{(Let u} = s^{-1}\lambda\text{)}$$

$$=u^3\left(\sum_{n=1}^{\infty}u^{n-1}\right)''=u^3\left(\frac{1}{1-u}\right)''=\frac{2u^3}{(1-u)^3}=\frac{2\lambda^3}{(s-\lambda)^3}$$

Similarly, $\mathcal{G}\{(n-1)(n-2)(n-3)\lambda^n\}=\dfrac{3!\lambda^4}{(s-\lambda)^4}$

In general, $\mathcal{G}\{(n-1)^{(p)}\lambda^n\}=\dfrac{p!\lambda^{p+1}}{(s-\lambda)^{p+1}}$

Note: All of the formulas above also apply when $\lambda=1$ and one doesn't "see" λ.

$$\mathcal{G}\{c_1a_n+c_2b_n\}=\sum_{n=1}^{\infty}\lambda^n(c_1a_n+c_2b_n)=c_1\sum_{n=1}^{\infty}\lambda^n a_n+c_2\sum_{n=1}^{\infty}\lambda^n b_n=c_1\mathcal{G}\{a_n\}+c_2\mathcal{G}\{b_n\}$$

Note: In the remaining theorems, $\mathcal{G}\{a_n\}=A(s)$.

$$\mathcal{G}\{a_{n+1}\}=\sum_{n=1}^{\infty}s^{-n}a_{n+1}=s\sum_{n=1}^{\infty}s^{-(n+1)}a_{n+1}=s\sum_{m=2}^{\infty}s^{-m}a_m=s\left[\sum_{m=1}^{\infty}s^{-m}a_m-s^{-1}a_1\right]=s\mathcal{G}\{a_n\}-a_1=sA-a_1$$

$$\mathcal{G}\{a_{n+2}\}=s\mathcal{G}\{a_{n+1}\}-a_2=s(sA-a_1)-a_2=s^2A-sa_1-a_2$$

$$\mathcal{G}\{a_{n+j}\}=s^jA-s^{j-1}a_1-s^{j-2}a_2-\ldots-sa_{j-1}-a_j$$

Sample Applications

1. Find an explicit formula for a_n if $a_{n+2} = a_{n+1} + 2a_n$ and $a_1 = 1,\ \ a_2 = 5$

$$\mathcal{G}\{a_{n+2}\} = \mathcal{G}\{a_{n+1}\} + 2\mathcal{G}\{a_n\}$$
$$s^2A - sa_1 - a_2 = sA - a_1 + 2A$$
$$s^2A - s - 5 = sA - 1 + 2A$$
$$\left(s^2 - s - 2\right)A = s + 4$$
$$A = \frac{s+4}{(s+1)(s-2)} = \frac{2}{s-2} - \frac{1}{s+1} \quad \text{(Partial fractions)}$$

$$a_n = 2^n + (-1)^n$$

2. Suppose the above problem were changed to $a_{n+2} = a_{n+1} + 2a_n + n^2 - 3n$ and $a_1 = 1,\ \ a_2 = 5$

$$\mathcal{G}\{a_{n+2}\} = \mathcal{G}\{a_{n+1}\} + 2\mathcal{G}\{a_n\} + \mathcal{G}\{(n-1)(n-2) - 2\}$$
$$s^2A - sa_1 - a_2 = sA - a_1 + 2A + \frac{2}{(s-1)^3} - \frac{2}{s-1}$$
$$s^2A - s - 5 = sA - 1 + 2A + \frac{2}{(s-1)^3} - \frac{2}{s-1}$$
$$s^2A - sA - 2A = s + 4 + \frac{2}{(s-1)^3} - \frac{2}{s-1}$$
$$A = \frac{(s-1)^3(s+4) + 2 - 2(s-1)^2}{(s+1)(s-2)(s-1)^3}$$
$$A = \frac{2}{s-2} + \frac{\frac{1}{4}}{s-1} + \frac{-\frac{5}{4}}{s+1} + \frac{-\frac{1}{2}}{(s-1)^2} + \frac{-1}{(s-1)^3}$$

Therefore

$$a_n = A^{-1} = 2^n + \frac{1}{4}(1)^n + \frac{5}{4}(-1)^n - \frac{1}{2}(n-1) - \frac{(n-1)(n-2)}{2}$$
$$= 2^n + \frac{5}{4}(-1)^n - \frac{1}{2}n^2 + n - \frac{1}{4}$$

3. Consider the following system of recursion equations

$$a_{n+1} = 5a_n + 2b_n \qquad a_1 = 1$$

$$b'_{n+1} = -6a_n - 3b_n \qquad b_1 = 3$$

Using transforms we get

$$sA - a_1 = 5A + 2B$$

$$sB - b_1 = -6A - 3B$$

It follows that

$$(s-5)A - 2B = 1$$

$$6A + (s+3)B = 3$$

After using Cramer's Rule we get

$$A = \frac{s+9}{s^2 - 2s - 3} = \frac{3}{s-3} + \frac{-2}{s+1}$$

$$B = \frac{3s-21}{s^2 - 2s - 3} = \frac{-3}{s-3} + \frac{6}{s+1}$$

Eventually

$$a_n = 3^n + 2(-1)^n$$

$$b_n = -(3)^n - 6(-1)^n$$